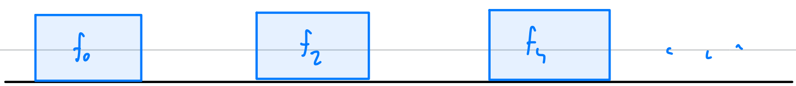
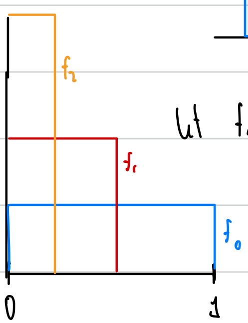
Measure Theory with Ergodic Horizons
Lecture 18

First we understand when pointwise convergence implies convergence in L^1-norm. We saw examples last time of pointwise convergence not implying L^1-convergence, we recall them here:

Examples.

(a) Let $f_n := \mathbb{1}_{[n,n+1)}$, then $f_n \to 0$ pointwise, but $\|0-f_n\|_1 = \|f_n\| = 1$, so $f_n \not\to_{L^1} 0$.
In fact, (f_n) doesn't converge in L^1 to anything since $\|f_n - f_m\| = 2$ $\forall n \neq m$.

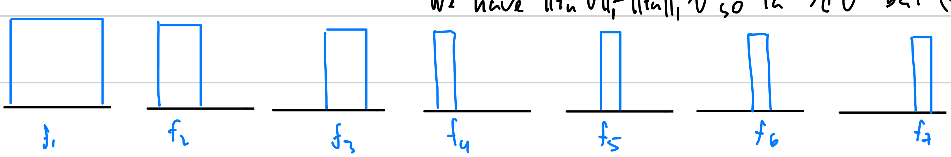




(b) Let $f_n := n\,\mathbb{1}_{(0,\frac{1}{n}]}$, $n \geq 1$. Then again $f_n \to 0$ pointwise, but $\|f_n - 0\|_1 = \|f_n\| = 1$ so $f_n \not\to_{L^1} 0$. Again, (f_n) doesn't converge in L^1 at all because $\|f_n - f_m\|_1 \geq 1$ for all $n \neq m$.

(c) $f_1 = \chi_{[0,1]}$, $f_2 = \chi_{[0,1/2]}$, $f_3 = \chi_{[1/2,1]}$, $f_4 = \chi_{[0,1/4]}$, $f_5 = \chi_{[1/4,1/2]}$, $f_6 = \chi_{[1/2,3/4]}$, $f_7 = \chi_{[3/4,1]}$, and in general, $f_n = \chi_{[j/2^k,(j+1)/2^k]}$ where $n = 2^k + j$ with $0 \leq j < 2^k$.

We have $\|f_n - 0\|_1 = \|f_n\|_1 \to 0$ so $f_n \to_{L^1} 0$ but (f_n) doesn't converge pointwise.



Note that the MCT already gives a sufficient condition for L^1-convergence:

Cor. For $f_n, f \in L^1(X,\mu)$, if $0 \leq f_n \nearrow f$ then $f_n \to_{L^1} f$.

Proof. $\|f-f_n\|_1 = \int |f-f_n| \, d\mu = \int (f-f_n) \, d\mu = \int f \, d\mu - \int f_n \, d\mu \xrightarrow{MCT} 0$. $\square$

What about the non-monotone sequences as in examples (a) and (b) above? Note that in these examples the whole sequence (f_n) was not under some "finite umbrella", i.e. not dominated by an integrable function. The following shows that once this is fixed, we get L^1-convergence.

Dominated Convergence Theorem (DCT). Let $f_n, f \in L^1(X,\mu)$ such that $\exists$ non-negative $g \in L^1(X,\mu)$ such that $|f_n| \leq g$ a.e. for all n. Then $f_n \to f$ a.e. $\Rightarrow f_n \to_{L^1} f$. In particular, $\int f_n \, d\mu \to \int f \, d\mu$.

Proof. We may assume the a.e. conditions hold everywhere by throwing out a null set. We only know MCT and Fatou's lemma. The first one is not applicable, and the second one is applicable to only non-negative functions. Applying Fatou to $|f_n|$ won't give anything useful about $\int f_n \, d\mu$ or $\int |f_n - f| \, d\mu$. We apply Fatou to $g \pm f_n$.

$$\cancel{\int g \, d\mu} - \int f \, d\mu = \int \liminf_n (g - f_n) \leq \liminf_n \int (g-f_n) \, d\mu = \cancel{\int g \, d\mu} - \limsup_n \int f_n \, d\mu$$

$$\cancel{\int g \, d\mu} + \int f \, d\mu = \int \liminf_n (g + f_n) \leq \liminf_n \int (g+f_n) \, d\mu = \cancel{\int g \, d\mu} + \liminf_n \int f_n \, d\mu$$

have $\limsup_n \int f_n \, d\mu \le \int f \, d\mu \le \liminf_n \int f_n \, d\mu$, so $\lim_n \int f_n \, d\mu = \int f \, d\mu$.
Applying what we've proven to the sequence $|f - f_n|$, since $|f-f_n| \le |f| + |f_n| \le 2g$ and $|f_n - f| \to 0$ pointwise, we get $\int |f_n - f| \, d\mu \to \int 0 \, d\mu = 0$, i.e. $f_n \to_{L^1} f$. $\square$

Remark. We didn't need to require f to be integrable as it is automatic. Indeed, $|f| = \lim_n |f_n| \le g$ so the monotonicity of the integral implies $\|f\|_1 = \int |f| \, d\mu \le \int g \, d\mu < \infty$.

Cor. Simple functions are dense in $L^1(X, \mu)$.

Cor. Let (X, μ) be a countably generated measure space, i.e. Meas_μ is countably generated mod null. Then $L^1(X, \mu)$ is separable.

Proof. The proof was a homework exercise for σ-finite (X, μ), in order to use the uniqueness part of Carathéodory's extension theorem. However, as suggested by Vahan, we don't need the whole space to be σ-finite because we only need to approximate finite measure sets. We just need the following better version of uniqueness in Carathéodory's extension theorem:

Carathéodory's extension (with stronger uniqueness). Let $\mathcal{A}$ be an algebra on a set X and let μ be a ctbly additive measure on $\mathcal{A}$ (premeasure). Then μ extends to a measure on $\langle \mathcal{A} \rangle_\sigma$, namely, the outer measure μ^*. Moreover, such an extension is unique for sets of finite outer measure, i.e. if ν is also an extension of μ to a measure on $\langle \mathcal{A} \rangle_\sigma$, then for each $B \in \langle \mathcal{A} \rangle_\sigma$, we have $\nu(B) \leq \mu^*(B)$, where equality holds if $\mu^*(B) < \infty$.

Proof of equality $\nu(B) = \mu^*(B)$. We've already shown that the extension is unique if μ is σ-finite and we will reduce to the case where μ is finite. Let $B \in \langle \mathcal{A} \rangle_\sigma$ be such that $\mu^*(B) < \infty$. Then there are sets $A_n \in \mathcal{A}$ such that $B \subseteq \tilde{X} := \bigcup_{n \in \mathbb{N}} A_n$ and $\sum_{n \in \mathbb{N}} \mu(A_n) < \infty$, and by disjointification, we may assume the A_n are pairwise disjoint. We then restrict to $\tilde{X}$ by observing that $\tilde{\mathcal{A}} := \{A \cap \tilde{X} : A \in \tilde{\mathcal{A}}\}$ is algebra and defining $\tilde{\mu} : \tilde{\mathcal{A}} \to [0, \infty]$ by $\tilde{\mu}(A \cap \tilde{X}) := \sum_{n \in \mathbb{N}} \mu(A \cap A_n)$. We can easily check that $\tilde{\mu} = \mu^*|_{\tilde{\mathcal{A}}} = \nu|_{\tilde{\mathcal{A}}}$, indeed, for each $A \in \mathcal{A}$, we have:

$$\tilde{\mu}(A \cap \tilde{X}) := \sum_{n \in \mathbb{N}} \mu(A \cap A_n) = \sum_{n \in \mathbb{N}} \mu^*(A \cap A_n) \text{ and } = \sum_{n \in \mathbb{N}} \nu(A \cap A_n).$$

But $\tilde{\mu}(\tilde{X}) < \infty$, so the original version of Carathéodory's extension gives that $\nu = \mu^*$ on $\langle \tilde{\mathcal{A}} \rangle_\sigma$, and in particular $\nu(B) = \mu^*(B)$ because $B \in \langle \tilde{\mathcal{A}} \rangle_\sigma$. □

Example. For $(\mathbb{R}^d, \lambda)$, $(2^{\mathbb{N}}, \mu_p)$, $(\mathbb{N}, \text{counting})$, the L^1 of these spaces is separable.

For a set X with counting measure μ, $L^1(X, \mu)$ is simply denoted by $\ell^1(X)$. In particular, $\ell^1(\mathbb{N})$ is the vector space of sequences (a_n) of reals that are absolutely summable, i.e. $\|(a_n)\|_1 := \sum_n |a_n| < \infty$, because for $f \in \ell^1(\mathbb{N})$,

$$\int f \, d\mu = \sum_{n \in \mathbb{N}} f(n).$$

Also, for $d \in \mathbb{N}$, i.e. $d = \{0, 1, \ldots, d-1\}$, what space is $\ell^1(d)$? Indeed,

$\ell^1(d) = \mathbb{R}^d$ with the $\|\cdot\|_1$-norm, i.e. $\|\vec{x}\|_1 := \sum_{n<d} |\vec{x}(n)|$ because for each $\vec{x} \in \ell^1(d)$, the integral is $\int \vec{x}(n)\, d\mu(n) = \sum_{n<d} \vec{x}(n)$.

We will now show that $\mathcal{L}^1(X,\mu)$ is always complete (as a pseudo-metric space), every Cauchy sequence has a limit. To show this it is convenient to use the following criterion of completeness for pseudo-normed vector spaces.

Completeness for normed vector-spaces. Let $(V, \|\cdot\|)$ be a real vector-space with a pseudo-norm $\|\cdot\|$. Then V is complete (as a pseudo-metric space) $\Leftrightarrow$ every absolutely convergent series $\sum_{n\in\mathbb{N}} f_n$ (i.e. $\sum_{n\in\mathbb{N}} \|f_n\| < \infty$) converges in norm (i.e. there is $f \in V$ such that $\|f - \sum_{n<N} f_n\| \to 0$ as $N \to \infty$).

Proof. $\Rightarrow$. Suppose that V is complete, i.e. every Cauchy sequence has a limit. Let $\sum_{n\in\mathbb{N}} f_n$ be an absolutely convergent series, i.e. $f_n \in V$ and $\sum_{n\in\mathbb{N}} \|f_n\| < \infty$. We want to show that $\sum_{n\in\mathbb{N}} f_n$ exists, i.e. that the partial sums $\sum_{n<N} f_n$ converge in norm. It's enough to check that the sequence of partial sums is Cauchy, but this is clear: for $N \leq M$

$$\left\|\sum_{n<N} f_n - \sum_{n<M} f_n\right\| = \left\|\sum_{N\leq n<M} f_n\right\| \overset{\Delta\text{-inequality}}{\leq} \sum_{N\leq n<M} \|f_n\| \leq \sum_{n\geq N} \|f_n\| \to 0 \text{ as } N \to \infty$$

because $\sum_{n\in\mathbb{N}} \|f_n\| < \infty$.

$\Leftarrow$. Suppose the right hand side and let (f_n) be a Cauchy sequence.

To use a property for series, we need to consider the series $\sum_{n\in\mathbb{N}}(f_{n+1}-f_n)$.
But this may not be absolutely convergent, i.e. it might be that $\sum_{n\in\mathbb{N}}\|f_{n+1}-f_n\|=\infty$.
We fix this by passing to a subsequence (f_{n_k}) so that $\|f_{n_{k+1}}-f_{n_k}\|\leq 2^{-k}$ which exists by the Cauchiness of the sequence. Then $\sum_{k\in\mathbb{N}}\|f_{n_{k+1}}-f_{n_k}\|<\infty$ so $\sum_{k\in\mathbb{N}}(f_{n_{k+1}}-f_{n_k})$ exists, i.e. the sequence of partial sums $\sum_{k<K} f_{n_{k+1}}-f_{n_k}$ $= f_{n_K}-f_{n_0}$ has a limit, hence (f_{n_k}) has a limit.
But a Cauchy sequence converges iff and only if it has a convergent subsequence, so (f_n) converges. $\square$

Theorem. For any measure space (X,μ), the normed-vector space $L^1(X,\mu)$ is complete.

Proof. We use the previous criterion: suppose $\sum_{n\in\mathbb{N}} f_n$ converges absolutely, i.e. $\sum_{n\in\mathbb{N}}\|f_n\|_1<\infty$. Need to show that $\sum_{n\in\mathbb{N}} f_n$ converges in L^1-norm. Let $g:=\sum_{n\in\mathbb{N}}|f_n|$.
By ctbl additivity of the integral (i.e. by MCT), we have

$$\|g\|_1=\int\sum_{n\in\mathbb{N}}|f_n|\,d\mu=\sum_{n\in\mathbb{N}}\int|f_n|\,d\mu=\sum_{n\in\mathbb{N}}\|f_n\|_1<\infty,$$

so $g\in L^1(X,\mu)$.
$\int\sum_n|f_n|\,d\mu<\infty$ implies in particular that $\sum_{n\in\mathbb{N}}|f_n|<\infty$ a.e. Thus, $f:=\sum_{n\in\mathbb{N}} f_n$ exists a.e., i.e. $\sum_{n<N} f_n\to f$ a.e. Moreover, $\int|f|\,d\mu=\int|\sum_n f_n|\,d\mu\leq$ $\leq\int\sum_n|f_n|\,d\mu=\|g\|_1<\infty$, so $f\in L^1(X,\mu)$.
Also, $|\sum_{n<N} f_n|\leq\sum_{n<N}|f_n|\leq\sum_{n\in\mathbb{N}}|f_n|=g$, so DCT gives $\|\sum_{n<N} f_n-f\|_1\to 0$, hence (f_n) converges to f in L^1-norm. $\square$